Math 107: Calculus II, Spring 2014: Midterm Exam I
Monday, March 2 2015

SOLUTIONS

1. (a) (10 points) Write the correct **form** of the partial fraction decomposition of the following rational function. **Do not** evaluate the undetermined coefficients.

$$\frac{x^3 + 2x - 5}{x^2(x^2 - 16)(x + 4)(x^2 + 4)}$$

Note that the denominator is not factored into irreducibles. In order to do partial fraction decomposition, the denominators needs to be written as a product of irreducible components (which are linear terms or irreducible quadratics).

$$x^2(x^2 - 16)(x + 4)(x^2 + 4) = x^2(x - 4)(x + 4)^2(x^2 + 4)$$

Thus, the correct partial fraction decomposition of the above rational function is

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 4} + \frac{D}{x + 4} + \frac{E}{(x + 4)^2} + \frac{Fx + G}{x^2 + 4}.$$

Note that the factor $x + 4$ that you factored out of $x^2 - 16$ combines with the other factor of $x + 4$, so the linear term $x + 4$ shows up with power 2 in the denominator and you need the terms

$$\frac{D}{x + 4} + \frac{E}{(x + 4)^2}$$

in the partial fraction decomposition to account for it. Note that writing

$$\frac{D}{x + 4} + \frac{E}{x + 4}$$

is NOT correct. You had to do some examples in a homework once to see why you really do need the iterated powers of each irreducible as the denominators of the fractions in the decomposition, all the way up to the power it appears with.

Note: This question was very similar to exercise 5 from the practice from Friday.

(b) (15 points) Compute the indefinite integral $\int \frac{2x^3+2}{x^2+4}dx$. **Show all your work!**

First note that the rational function is improper, so do long division to get

$$\frac{2x^3+2}{x^2+4} = 2x + \frac{(-8x+2)}{x^2+4}.$$

So

$$\begin{aligned} \int \frac{2x^3+2}{x^2+4}dx &= \int 2xdx + \int \frac{-8x}{x^2+4}dx + \int \frac{2}{x^2+4}dx \\ &= x^2 - 4\ln|x^2+4| + \tan^{-1}(\frac{x}{2}) + C. \end{aligned}$$

For the middle integral we did an easy u- substitution with $u = x^4+4$ and $du = 2xdx$, and recognized that $-4\int \frac{1}{u}du = -4\ln|u| + C$.

For the last integral we rewrote $\frac{2}{x^2+4} = \frac{2}{4((\frac{x}{2})^2+1)}$ and did an easy u- substitution with $u = \frac{x}{2}$ and $du = \frac{1}{2}dx$, and recognized that $\int \frac{1}{u^2+1}du = \tan^{-1}(u) + C$.

Note that this question was on your homework !! I just multiplied the question you had to do not he homework by 2, which actually makes the constants come out nicer.

2. (a) (12 points) Compute the improper integral $\int_1^\infty \frac{dx}{x^2}$. **Show all your work!**

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx \\ &= \lim_{b\to\infty}(-\frac{1}{x})|_1^b \\ &= \lim_{b\to\infty}(1-\frac{1}{b}) = 1.\end{aligned}$$

Note that you had to evaluate this as a LIMIT. You cannot plug in ∞ and evaluate there as if it were a number.

Note that you had to do precisely this example in class, and then I did the one for any power p for you. This was similar to the practice question 4 also, but much easier.

(b) (12 points) Does the improper integral $\int_1^\infty \frac{dx}{\sqrt{3+x^4}}$ converge or diverge? **Show all your work!**

Note that

$$\sqrt{3+x^4} > \sqrt{x^4} = x^2.$$

Therefore,

$$\frac{1}{\sqrt{3+x^4}} < \frac{1}{x^2},$$

and $\int_1^\infty \frac{dx}{\sqrt{3+x^4}}$ converges by comparison with $\int_1^\infty \frac{dx}{x^2}$, which we know to converge by part (a).

Note that this is just like the example we did in class that I also wrote a note on. You had similar comparisons on the homework and some harder ones, too.

3. (21 points) Are the following statements true or false? Write **clearly** either **TRUE** or **FALSE** in the answer lines. Write the **whole word** and **not just T or F,** because T and F can look similar.

No justification is needed! For some of the questions, you might want to do some computations to figure out the answer. **The next page has intentionally been left blank for scratch work for this question**. However, **only the answer of True or False** will be graded.

(a) There are more combinations of n objects taken k at a time than there are permutations of n objects taken k at a time (assuming $k > 1$).
FALSE
There are more permutations, because for permutations we count all the different orderings of the k objects. More precisely,

$$C(n,k) = \frac{P(n,k)}{k!},$$

so the number of combinations is the number of permutations divided by something.

(b) For any events A and B of a sample space, we have an equality of conditional probabilities $P(A \mid B) = P(B \mid A)$.
FALSE
What is true is that $P(A \mid B)P(B) = P(B \mid A)P(A)$.
However, we saw in a lot of examples that $P(A \mid B) = P(B \mid A)$ is not true. For instance, in all the questions you use Bayes's formula for, you usually know of one $P(A \mid B)$ and $P(B \mid A)$ and need to compute the other.

(c) Suppose we toss 2 fair dice. Let A be the event that the first die is 3 and B the event that the sum of the dice is 6. The events A and B are independent.
FALSE
Note that $P(A \cap B) = \frac{1}{36}$ since there is only one outcome out of the possible 36 ones in which the first die is 3 and the sum is 6.

Now $P(A) = \frac{1}{6}$ because there are 6 outcomes of the 36 possible ones that have 3 on the first die.

And $P(B) = \frac{5}{36}$ since there are 5 ways to get sum 6: $(1,5),(2,4),(3,3),(4,2),(5,1)$.

Thus $P(A \cap B) \neq P(A)P(B)$.

(d) Suppose we toss 2 fair dice. Let A be the event that the first die is 3 and B the event that the sum of the dice is 7. The events A and B are independent. TRUE
Note that $P(A \cap B) = \frac{1}{36}$ since there is only one outcome out of the possible 36 ones in which the first die is 3 and the sum is 6.

Now $P(A) = \frac{1}{6}$ because there are 6 outcomes of the 36 possible ones that have 3 on the first die.

But now $P(B) = \frac{6}{36}$ since there are 6 ways to get sum 6: $(1,6),(2,5),(3,4),(4,3)(5,2),(6,1)$.

Thus $P(A \cap B) = P(A)P(B)$.

Note that c and d were very similar to practice question 3 (and the suggested homework problems for last week.)

Note for the following questions: The graphs of x^5 and of $\frac{1}{x^5}$ are both symmetric around the origin.

(e) The improper integral $\displaystyle\int_{-\infty}^{\infty} x^5 dx$ converges to 0. FALSE

Each of $\int_{-\infty}^{0} x^5 dx$ and $\int_{0}^{\infty} x^5 dx$ diverges to ∞, and $-\infty$, respectively. So this integral diverges.

Note that this you and to do as a clicker question ones with x^3 instead of x^5 and we discussed it in class.

(f) The integral $\displaystyle\int_{-1}^{1} \frac{dx}{x^5}$ is equal to 0. FALSE

This is an improper integral because it has an infinite discontinuity at 0 and each of $\int_{-} 1^0 \frac{1}{x^5} dx$ and $\int_0^1 \frac{1}{x^5} dx$ diverges. So this integral diverges.

Note that showing that $\int_0^1 \frac{1}{x^5}$ diverges was part of practice question 4. And you had to do this integral in general for any power p on the homework.

(g) The integral $\displaystyle\int_{-1}^{1} x^5 dx$ is equal to 0.

TRUE

This integral is not improper, and it's an odd function with a symmetric graph about the origin integrated over a symmetric interval around the origin. So the areas cancel to 0. It's also easy to see directly by evaluating that it's 0.

4. (a) (15 points) Two cards are chosen at random from a pack of 52 playing cards. What is the probability that at least one of them is a face card (Jack, Queen or King)?

(Carry out the computation until the end to get the answer.)

Show all your work!

Note that there are 12 face cards and 40 non-face cards.

Let

$$E = \text{event that at least one is a face card.}$$

Let's compute

$$E^c = \text{event that neither is a face card}$$

instead and then use

$$P(E) = 1 - P(E^c).$$

Let A be event that the first card is a face card and B be event that the second card is a face card. The event E^c is the event that the first card is not a face card and the second card is not a face card, so

$$E^c = A^c \cap B^c.$$

Note that A^c and B^c are dependent, and

$$P(A^c) = \frac{40}{52} = \frac{10}{13} \qquad \text{and} \qquad P(B^c \mid A^c) = \frac{39}{51} = \frac{13}{17}.$$

Thus

$$P(A^c \cap B^c) = \frac{10}{13} \times \frac{13}{17} = \frac{10}{17}.$$

Thus

$$P(E) = 1 - \frac{10}{17} = \frac{7}{17}.$$

Note that was just the same with practice question 6 and the question from class that I wrote 3 different solutions to – you could've written a different solution to this to, but they should all give you the same final answer.

(b) (15 points) Suppose that half of the people in the class like math and half of the people do not like math. A test has been created to determine who likes math and who does not. However the test is not completely accurate: only 90% of the people who like math pass the test, and 30% of the people who do not like math manage to pass the test anyway.

If you have passed the test, what is the probability that you actually like math? (Carry out the computation until the end to get the answer.)

Show all your work!

Let A be the event that one likes math and A^c the event that one doesn't– these partition the sample space. We know $P(A) = P(A^c) = \frac{1}{2}$. Let B be the event that one passes the test.

We are given the conditional probabilities $P(B \mid A) = \frac{9}{10}$ and $P(B \mid A^c) = \frac{3}{10}$. We have to compute the conditional probability $P(A \mid B)$ of liking math *if* we have passed the test.

Using Bayes, we compute

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} \\ &= \frac{(\frac{1}{2})(\frac{9}{10})}{(\frac{1}{2})(\frac{9}{10}) + (\frac{1}{2})(\frac{3}{10})} \\ &= \frac{(\frac{1}{2})(\frac{9}{10})}{(\frac{1}{2})(\frac{9}{10} + \frac{3}{10})} \\ &= \frac{\frac{9}{10}}{\frac{12}{10}} = \frac{9}{12} = \frac{3}{4}. \end{aligned}$$

Note that you were supposed to set the events and say what conditional probability you are computing. Most of you have done so, and it was a pleasure to grade.

Note that this question was almost the same with the many practice questions on Bayes.